

# 3-Dimensional Schlaefli Formula and Its Generalization

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*Dedicated to the memory of Xiao-Song Lin*

**Abstract.** Several identities similar to the Schlaefli formula are established for tetrahedra in a space of constant curvature.

Keywords: tetrahedron, dihedral angles, volume, lengths, and the cosine law.

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## 1 Introduction

One of the most important identities in low-dimensional geometry is the Schlaefli formula. It states that for a tetrahedron in a constant curvature  $\lambda = \pm 1$  space, the volume  $V$ , the length  $x_{ij}$ , and the dihedral angle  $a_{ij}$  at the  $ij$ -th edge are related by

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$$\frac{\partial V}{\partial a_{ij}} = \frac{\lambda}{2} x_{ij} \quad (1.1)$$

where  $V = V(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34})$  is a function of the angles. See for instance [5] or [1] for a proof.

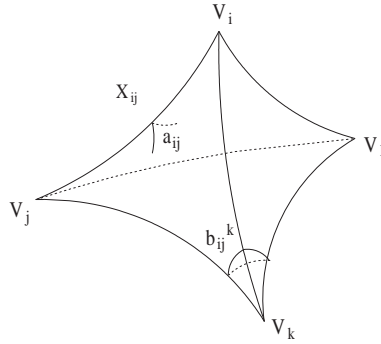


Figure 1.1

In 3-dimensional polyhedral geometry, a space is obtained by isometrically gluing tetrahedra along their codimension-1 faces. The metric is determined by the edge lengths and the curvature at an edge is  $2\pi$  less the sum of dihedral angles at the edge. From this point of view, the Schlaefli formula relates the most important geometric quantities: the volume, the metric (=edge lengths) and the curvature (=dihedral angles) in a simple elegant identity. The Schlaefli formula plays a vital role in a variational principle for triangulated 3-manifolds. See for instance Regge's work [6] on discrete general relativity.

One consequence of (1.1) is that differential 1-forms

$$\sum x_{ij} da_{ij} \quad \text{and} \quad \sum a_{ij} dx_{ij} \quad (1.2)$$

are closed.

We may recover the volume function  $V$  in (1.1) by integrating the 1-form  $\sum x_{ij} da_{ij}$ . Thus the closeness of the 1-forms in (1.2) essentially captures the Schlaefli formula.

The basic problem in polyhedral geometry is to understand the relationship between the metric and its curvature. In the case of tetrahedra, this prompts us to study the curvature map  $K(x) = a$  sending the edge length  $x = (x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34})$  to the dihedral angle  $a = (a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34})$ . The Jacobian matrix  $D(K)$  of the curvature map is the  $6 \times 6$  matrix  $[\frac{\partial a_{ij}}{\partial x_{rs}}]_{6 \times 6}$ . The closeness of the 1-forms in (1.2) is equivalent to say that the Jacobian matrix  $D(K)$  is symmetric. It turns out the Jacobian matrix  $[\frac{\partial a_{ij}}{\partial x_{rs}}]_{6 \times 6}$  enjoys many more symmetries. One of the symmetry was discovered by E. Wigner [9] and Taylor-Woodward [8]. The purpose of this paper is to find the com-

plete set of symmetries of the Jacobian matrix  $D(K)$  of the curvature map. These symmetries should have applications in 3-dimensional topology and geometry. In particular, the relationships between the Jacobian matrix  $D(K)$ , the 6j symbols, the quantum 6j symbols, and the volume conjecture are very attractive problems. See for instance the work of [8] and [7].

The complete set of symmetries was discovered by us a few years ago. We thank Walter Neumann for suggesting us to write it up for publication.

This paper is dedicated to the memory of Xiao-Song Lin who made important contributions to low-dimensional topology. He was a great colleague and friend.

The paper is organized as follows. In §2, we state the main theorem. These theorems are proved in §3. A more general version of it involving complex valued lengths and angles can be found in [4].

## 2 The main theorem

Let a tetrahedron in  $\mathbf{S}^3$  or  $\mathbf{H}^3$  or  $\mathbf{E}^3$  have vertices  $v_1, v_2, v_3, v_4$ . Let  $a_{ij}$  and  $x_{ij}$  be the dihedral angle and the edge length at the  $ij$ -th edge  $v_i v_j$ . We consider the angle  $a_{ij}$  as a function of the lengths  $x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}$ .

Here is our main result.

**Theorem 2.1.** *Define  $P_{rs}^{ij} = \frac{1}{\sin(a_{ij})\sin(a_{rs})} \frac{\partial a_{ij}}{\partial x_{rs}}$ . Then these quantities satisfy the following identities for any tetrahedron in any of the three geometries  $\mathbf{S}^3, \mathbf{H}^3, \mathbf{E}^3$ . The indices  $i, j, k, l$  are assumed to be pairwise distinct.*

- (i) (Schlaefli)  $P_{rs}^{ij} = P_{ij}^{rs}$ .
- (ii) ([9], [8])  $P_{kl}^{ij} = P_{jl}^{ik} = P_{jk}^{il}$ .
- (iii)  $P_{ik}^{ij} = -P_{kl}^{ij} \cos a_{jk}$ .
- (iv)  $P_{ij}^{ij} = P_{kl}^{ij} w_{ij}$  where  $w_{ij} = (c_{ij}c_{jk}c_{ki} + c_{ij}c_{jl}c_{li} + c_{ik}c_{jl} + c_{il}c_{jk}) / \sin^2(a_{ij})$  and  $c_{rs} = \cos(a_{rs})$ .
- (v)  $P_{rs}^{ij} = P_{r's'}^{i'j'}$  where  $\{i, j\} \neq \{r, s\}$  and for a subset  $\{a, b\} \subset \{i, j, k, l\}$ , the set  $\{a', b'\}$  is  $\{i, j, k, l\} - \{a, b\}$ .

In the spaces  $\mathbf{S}^3$  and  $\mathbf{H}^3$  of constant curvature  $\lambda = \pm 1$ , a tetrahedron is determined by its dihedral angles  $a_{ij}$ . Thus the length  $x_{ij}$  can be considered as a function of the angles. The similar theorem is,

**Theorem 2.2.** *Define  $R_{rs}^{ij} = \frac{1}{\sin(\sqrt{\lambda}x_{ij})\sin(\sqrt{\lambda}x_{rs})} \frac{\partial x_{ij}}{\partial a_{rs}}$ . Then these quantities satisfy the following identities for any tetrahedron in spherical and hyperbolic geometries. Let the indices  $i, j, k, l$  be distinct.*

- (i) (Schlaefli)  $R_{rs}^{ij} = R_{ij}^{rs}$ .

- (ii) ( $[9]$ ,  $[8]$ )  $R_{kl}^{ij} = R_{jl}^{ik} = R_{jk}^{il}$ .
- (iii)  $R_{ik}^{ij} = R_{kl}^{ij} \cos(\sqrt{\lambda} x_{il})$ .
- (iv)  $R_{ij}^{ij} = R_{kl}^{ij} w_{ij}$  where

$$w_{ij} = \frac{-c_{ij}c_{ik}c_{il} - c_{ji}c_{jk}c_{jl} + c_{ik}c_{jl} + c_{il}c_{jk}}{\sin^2(\sqrt{\lambda} x_{ij})},$$

and  $c_{rs} = \cos(\sqrt{\lambda} x_{rs})$ ,

- (v)  $R_{rs}^{ij} = R_{r's'}^{i'j'}$  where  $\{i, j\} \neq \{r, s\}$  and for a subset  $\{a, b\} \subset \{i, j, k, l\}$ , the set  $\{a', b'\}$  is  $\{i, j, k, l\} - \{a, b\}$ .

We remark that the matrices  $[\frac{\partial a_{ij}}{\partial x_{rs}}]$  and  $[\frac{\partial x_{ij}}{\partial a_{rs}}]$  are inverse of each other when  $\lambda \neq 0$ . Theorem 2.2 follows from theorem 2.1 by taking the dual. Indeed, in the spherical tetrahedral case, the dual tetrahedron has dihedral angle  $\pi - x_{ij}$  and edge length  $\pi - a_{ij}$  at the  $kl$ -th edge of the dual simplex. Thus, theorem 2.2 follows. The hyperbolic tetrahedra case in theorem 2.2 can be deduced from spherical case by analytical continuation.

Theorem 2.1 suggests that the matrix  $MD(K)M = [P_{rs}^{ij}]_{6 \times 6}$  where  $M$  is the diagonal matrix whose diagonal entries are  $\frac{1}{\sin(a_{ij})}$  exhibits more symmetries than the Jacobian matrix  $D(K)$ .

Both theorems are special cases of a complex valued edge-length and dihedral angle relation. This will be discussed in [4].

### 3 A proof of theorem 2.1

We need to recall the cosine law and its derivative form in order to compute the Jacobian matrix  $[\frac{\partial a_{ij}}{\partial x_{rs}}]$  effectively.

Let  $K^2 = \mathbf{S}^2$ , or  $\mathbf{H}^2$  or  $\mathbf{E}^2$  be the space of constant curvature  $\lambda = 1, -1$ , or  $0$ . Define a function  $S_\lambda(t)$  as follows.  $S_0(t) = t$ ;  $S_1(t) = \sin(t)$  and  $S_{-1}(t) = \sinh(t)$ . The sine law for a triangle of lengths  $l_1, l_2, l_3$  and opposite angles  $a_1, a_2, a_3$  in  $K^2$  can be stated as

$$\frac{S_\lambda(l_i)}{\sin(a_i)} = \frac{S_\lambda(l_j)}{\sin(a_j)} \quad (3.1)$$

A different way to state the sine law is that the expression

$$A_{ijk} = \sin(a_i)S_\lambda(l_j)S_\lambda(l_k)$$

is symmetric in indices  $i, j, k$  where  $\{i, j, k\} = \{1, 2, 3\}$ . For this reason, we call  $A_{ijk} = A_{123}$  the  $A$ -invariant of the triangle.

**Proposition 3.1.** ([2], [3]) *Let a triangle in  $K^2$  have inner angles  $a_1, a_2, a_3$  and edge lengths  $l_1, l_2, l_3$  so that  $l_i$ -th edge is opposite to the angle  $a_i$ . Then*

- (i)  $\frac{\partial a_i}{\partial l_j} = -\frac{\partial a_i}{\partial l_i} \cos(a_k)$  where  $\{i, j, k\} = \{1, 2, 3\}$ ,  
(ii)  $\frac{\partial a_i}{\partial l_i} = \frac{S_\lambda(l_i)}{A_{123}}$

See [2] or [3] for a proof.

Let us introduce some notations before beginning the proof. The indices  $i, j, k, l$  are pairwise distinct, i.e.,  $\{i, j, k, l\} = \{1, 2, 3, 4\}$ . The face triangle  $\Delta v_i v_j v_k$  will be denoted  $\Delta ijk$ . The inner angle at the vertex  $v_k$  of the triangle  $\Delta ijk$  is denoted by  $b_{ij}^k$ . The link at the vertex  $v_k$ , denoted by  $Lk(v_k)$ , is a spherical triangle with edge lengths  $b_{ji}^k, b_{il}^k, b_{lj}^k$  and inner angles  $a_{ki}, a_{kj}, a_{kl}$  so that  $a_{ki}$  is opposite to  $b_{jl}^k$ . The A-invariant of the triangle  $\Delta ijk$  is denoted by  $A_{ijk}$ .

In the calculation below, we consider  $b_{jk}^i$  as a function of  $x_{rs}$ 's using the cosine law for the triangle  $\Delta ijk$ . By the definition, we have,

$$\frac{\partial b_{jk}^i}{\partial x_{rs}} = 0 \quad (3.2)$$

if  $\{r, s\}$  is not a subset of  $\{i, j, k\}$ . The function  $a_{ij}$  is considered as a function of  $b_{st}^r$ 's by the cosine law applied to either the link  $Lk(v_i)$  or  $Lk(v_j)$ . In this way the dihedral angle  $a_{ij}$ , when considered as a function of the lengths  $x_{rs}$ 's, is a composition function.

To prove theorem 2.1, note that identity (i) in theorem 2.1 is the Schlaefli formula (1.2). Identity (v) follows from identity (iii). By symmetry, we only need to consider three partial derivatives:  $\frac{\partial a_{ij}}{\partial x_{kl}}, \frac{\partial a_{ij}}{\partial x_{ik}}$  and  $\frac{\partial a_{ij}}{\partial x_{ij}}$ .

### 3.1 The partial derivatives $\frac{\partial a_{ij}}{\partial x_{ik}}$ and $\frac{\partial a_{ij}}{\partial x_{kl}}$

Consider the link  $Lk(v_i)$ . Using proposition 3.1(ii), the chain rule and (3.2), we have (see Fig. 2.1(a)),

$$\frac{\partial a_{ij}}{\partial x_{kl}} = \frac{\partial a_{ij}}{\partial b_{kl}^i} \frac{\partial b_{kl}^i}{\partial x_{kl}} = \frac{\partial a_{ij}}{\partial b_{kl}^i} \frac{S_\lambda(x_{kl})}{A_{ikl}}. \quad (3.3)$$

Similarly, using  $Lk(v_j)$ , we have

$$\frac{\partial a_{ij}}{\partial x_{kl}} = \frac{\partial a_{ij}}{\partial b_{kl}^j} \frac{\partial b_{kl}^j}{\partial x_{kl}} = \frac{\partial a_{ij}}{\partial b_{kl}^j} \frac{S_\lambda(x_{kl})}{A_{jkl}}. \quad (3.4)$$

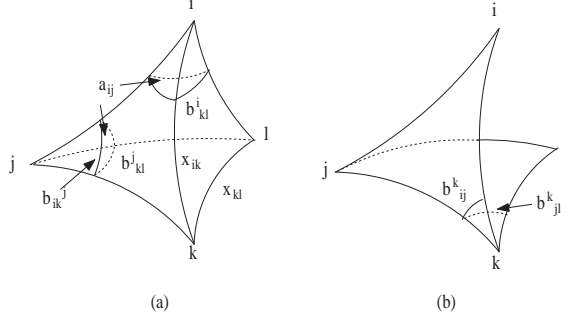


Figure 2.1

Now we use the link  $\text{Lk}(v_j)$  to find  $\frac{\partial a_{ij}}{\partial x_{ik}}$ . By (3.2) and the chain rule, we have

$$\frac{\partial a_{ij}}{\partial x_{ik}} = \frac{\partial a_{ij}}{\partial b_{ik}^j} \frac{\partial b_{ik}^j}{\partial x_{ik}}. \quad (3.5)$$

By proposition 3.1 applied to  $\text{Lk}(v_j)$  and  $\Delta ijk$ , we see (3.5) is equal to,

$$-\frac{\partial a_{ij}}{\partial b_{kl}^j} \cos(a_{jk}) \frac{S_\lambda(x_{ik})}{A_{ijk}}. \quad (3.6)$$

Using (3.4), we can write (3.6) as,

$$-\frac{\partial a_{ij}}{\partial x_{kl}} \cos(a_{jk}) \frac{A_{jkl} S_\lambda(x_{ik})}{A_{ijk} S_\lambda(x_{kl})}. \quad (3.7)$$

Now by the definition of the A-invariant of triangles  $\Delta ijk$  and  $\Delta jkl$  (see Fig. 2.1(b)), we have,

$$A_{jkl} = S_\lambda(x_{jk}) S_\lambda(x_{kl}) \sin(b_{jl}^k) \quad \text{and} \quad A_{ijk} = S_\lambda(x_{jk}) S_\lambda(x_{ik}) \sin(b_{ij}^k). \quad (3.8)$$

Thus (3.7) can be simplified to

$$-\frac{\partial a_{ij}}{\partial x_{kl}} \cos(a_{jk}) \frac{\sin(b_{jl}^k)}{\sin(b_{ij}^k)}. \quad (3.9)$$

By the sine law applied to the spherical triangle  $\text{Lk}(v_k)$ , we see (3.9) is equal to

$$\frac{\partial a_{ij}}{\partial x_{ik}} = -\frac{\partial a_{ij}}{\partial x_{kl}} \cos(a_{jk}) \frac{\sin(a_{ik})}{\sin(a_{kl})}. \quad (3.10)$$

This is equivalent to identity (iii),

$$P_{ik}^{ij} = -P_{kl}^{ij} \cos a_{jk}. \quad (3.11)$$

Use the Schlaefli formula that  $P_{ik}^{ij} = P_{ij}^{ik}$ , we obtain from (3.11)

$$-P_{kl}^{ij} \cos(a_{jk}) = -P_{jl}^{ik} \cos(a_{jk}).$$

This shows that

$$P_{kl}^{ij} = P_{jl}^{ik}. \quad (3.12)$$

By symmetry, identity (ii) holds for all indices.

### 3.2 The partial derivative $\frac{\partial a_{ij}}{\partial x_{ij}}$

By (3.2), the chain rule, we have, in the triangle  $\text{Lk}(v_i)$ ,

$$\frac{\partial a_{ij}}{\partial x_{ij}} = \frac{\partial a_{ij}}{\partial b_{jk}^i} \frac{\partial b_{jk}^i}{\partial x_{ij}} + \frac{\partial a_{ij}}{\partial b_{jl}^i} \frac{\partial b_{jl}^i}{\partial x_{ij}}. \quad (3.13)$$

Using proposition 3.1, we see that (3.13) is equal to

$$\frac{\partial a_{ij}}{\partial b_{kl}^i} \cos(a_{ik}) \cos(b_{ik}^j) \frac{S_\lambda(x_{jk})}{A_{ijk}} + \frac{\partial a_{ij}}{\partial b_{kl}^i} \cos(a_{il}) \cos(b_{il}^j) \frac{S_\lambda(x_{jl})}{A_{ijl}}. \quad (3.14)$$

Using (3.3), we see (3.14) is equal to

$$\frac{\partial a_{ij}}{\partial x_{kl}} [\cos(a_{ik}) \cos(b_{ik}^j) \frac{S_\lambda(x_{jk}) A_{ikl}}{S_\lambda(x_{kl}) A_{ijk}} + \cos(a_{il}) \cos(b_{il}^j) \frac{S_\lambda(x_{jl}) A_{ikl}}{S_\lambda(x_{kl}) A_{ijl}}]. \quad (3.15)$$

Using the sine law for triangles  $\Delta ikl$  and  $\Delta ij l$  as in (3.8), we can rewrite (3.15) as

$$\frac{\partial a_{ij}}{\partial x_{kl}} [\cos(a_{ik}) \cos(b_{ik}^j) \frac{\sin(b_{il}^k)}{\sin(b_{ij}^k)} + \cos(a_{il}) \cos(b_{il}^j) \frac{\sin(b_{ik}^l)}{\sin(b_{ij}^l)}]. \quad (3.16)$$

Using the sine law in triangles  $\text{Lk}(v_k)$  and  $\text{Lk}(v_l)$ , we see that (3.16) is the same as

$$\frac{\partial a_{ij}}{\partial x_{kl}} [\cos(a_{ik}) \cos(b_{ik}^j) \frac{\sin(a_{kj})}{\sin(a_{kl})} + \cos(a_{il}) \cos(b_{il}^j) \frac{\sin(a_{lj})}{\sin(a_{lk})}]. \quad (3.17)$$

$$= P_{kl}^{ij} [\cos(a_{ik}) \cos(b_{ik}^j) \sin(a_{kj}) \sin(a_{ij}) + \cos(a_{il}) \cos(b_{il}^j) \sin(a_{lj}) \sin(a_{ij})]. \quad (3.18)$$

On the other hand, by the cosine law for the spherical triangle  $\text{Lk}(v_j)$ , we have

$$\cos(b_{ik}^j) \sin(a_{kj}) \sin(a_{ij}) = \cos a_{kj} \cos a_{ij} + \cos a_{lj}.$$

and

$$\cos(b_{il}^j) \sin(a_{lj}) \sin(a_{ij}) = \cos a_{jl} \cos a_{ij} + \cos a_{jk}.$$

Substitute these into (3.18), we obtain

$$\frac{\partial a_{ij}}{\partial x_{ij}} = P_{kl}^{ij} [c_{ij}c_{jk}c_{ki} + c_{ij}c_{jl}c_{li} + c_{ik}c_{jl} + c_{il}c_{jk}]$$

where  $c_{rs} = \cos(a_{rs})$ .

This is the identity (iv) since  $P_{ij}^{ij} = \frac{1}{\sin^2(a_{ij})} \frac{\partial a_{ij}}{\partial x_{ij}}$ .

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